Journal of Approximation Theory **119**, 127–131 (2002)
doi:10.1006/jath.2002.3701

NOTE

Density of Extremal Sets in Multivariate Chebyshev Approximation[1]

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Communicated by Manfred v. Golitschek

Received June 18, 2001; accepted April 1, 2002

There are numerous results concerning the density of extremal sets (points of maximal deviation) in univariate Chebyshev approximation. In this note, we show that in multivariate setting this density is preserved in some weak sense.

Let Ω_j, $j \in \mathbb{N}$, be finite subsets of $\mathbb{N}^2$ such that $\Omega_j \subset \Omega_{j+1}$ and $\bigcup_{n=1}^{\infty} \Omega_n = \mathbb{N}^2$. Consider the corresponding spaces

$$P(\Omega_n) := \left\{ p(\mathbf{z}) = \sum_{\mathbf{k} \in \Omega_n} a_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} \colon a_{\mathbf{k}} \in \mathbb{R} \right\}, \qquad n \in \mathbb{N},$$

of bivariate polynomials of variable $\mathbf{z} = (x, y) \in \mathbb{R}^2$.

Furthermore, with $I = [-1, 1]$ and any $f \in C(I^2)$ set

$$\|f\| = \max_{\mathbf{z} \in I^2} |f(\mathbf{z})|, \qquad E(f, \Omega_n) := \inf_{p \in P(\Omega_n)} \|f - p\|,$$

$$B(f, \Omega_n) := \{ p \in P(\Omega_n) \colon \|f - p\| = E(f, \Omega_n) \},$$

$$A(f, p) = \{ \mathbf{z} \in I^2 \colon |f - p|(\mathbf{z}) = E(f, \Omega_n) \}, \qquad p \in B(f, \Omega_n).$$

Hence $E(f, \Omega_n)$ is the distance from f to $P(\Omega_n)$, $B(f, \Omega_n)$ denotes the set of its best approximants in $P(\Omega_n)$, and $A(f, p)$ consists of points of maximal deviation from f to its best approximant $p \in B(f, \Omega_n)$.

[1]Supported by the OTKA Grant No. T034531.

0021-9045/02 $35.00

In the univariate case by a well-known result of Kadec (see [2, pp. 4–8]) the sets of maximal deviation are dense in the underlying interval. The following example shows that in the bivariate case these extremal sets can all belong to a linear segment in I^2. For $\mathbf{m} = (r, s) \in \mathbb{N}^2$ we set $D(\mathbf{m}) := \{(k, l) \in \mathbb{N}^2 \colon k \leqslant r, l \leqslant s\}$.

EXAMPLE. Let $D(n, 1) \subset \Omega_n \subset D(n, n)$, $g(x, y) = (y+1) f(x)$, where $f \in C(I)$. Denote by p_n^* the best approximant of f by univariate polynomials of degree $\leqslant n$. Then $\tilde{p}_n(x, y) := (y+1) p_n^*(x) \in B(g, \Omega_n)$, $(x, y) \in I^2$. Moreover for any $(x, y) \in A(g, \tilde{p}_n)$ we have $y = 1$. Thus for each $n \in \mathbb{N}$, there is a selection of best approximant from $B(g, \Omega_n)$ so that the corresponding extremal sets belong to the segment $\{(x, 1) \colon x \in I\} \subset I^2$.

By the above example the density may occur just in "one of the coordinates." Let us verify now that this "weak density" holds in general, in case of bivariate approximation on a square. (We consider only the two-dimensional case for the sake of convenience, the case of approximation on a d-dimensional cube is similar.)

We shall require that Ω_n, $n \in \mathbb{N}$, satisfies the following mild restrictions:

(i) if $\mathbf{m} \in \Omega_n$ then $D(\mathbf{m}) \subset \Omega_n$;
(ii) $\mathbf{m}_1 \notin D(\mathbf{m}_2)$ whenever $\mathbf{m}_1, \mathbf{m}_2 \in \Omega_{n+1} \setminus \Omega_n$;
(iii) $\frac{1}{\log n} \min\{r + s \colon (r, s) \in \mathbb{N}^2 \setminus \Omega_n\} \to \infty$, $n \to \infty$.

(Conditions (i)–(iii) hold for instance when $\Omega_n := \{(r, s) \colon r + s \leqslant n\}$.)

Furthermore for any $\mathbf{K} \subset \mathbb{R}^2$ denote by $Cl(\mathbf{K})$ its closure, and

$$\mathbf{K}^x := \{x \in \mathbb{R} \colon (x, y) \in \mathbf{K}\}, \qquad \mathbf{K}^y := \{y \in \mathbb{R} \colon (x, y) \in \mathbf{K}\}.$$

THEOREM. *Let $f \in C(I^2)$ and assume that Ω_n satisfies* (i)–(iii), $n \in \mathbb{N}$. *For any $p_n \in B(f, \Omega_n)$ and $n_0 \in \mathbb{N}$ set $\mathbf{A}_f := Cl(\bigcup_{n=n_0}^{\infty} \mathbf{A}(f, p_n))$. Then either $\mathbf{A}_f^x = I$ or $\mathbf{A}_f^y = I$.*

Thus, the projection of extremal sets to at least one of the axes must be dense. (The previous example shows that one cannot expect in general a stronger result.)

We shall need a lemma from [3, p. 36].

LEMMA. *Let $\Omega \subset \mathbb{N}^2$ be finite, and assume that $\mathbf{r} = (i, j) \in \Omega$ is such that $\mathbf{r} \notin D(\mathbf{s})$ whenever $\mathbf{s} \in \Omega$, $\mathbf{s} \neq \mathbf{r}$. Then for any $p(\mathbf{z}) = \sum_{\mathbf{k} \in \Omega} a_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$ we have $|a_{\mathbf{r}}| \leqslant 2^{i+j-1} \|p\|$.*

Proof of theorem. Assume that to the contrary there exist nonempty open intervals $T_1, T_2 \subset I$ such that $x \notin T_1$, $y \notin T_2$ whenever $(x, y) \in \mathbf{A}_f$. Since the Chebyshev constant of the sets $I \setminus T_1$ and $I \setminus T_2$ is less than $\frac{1}{2}$ (see, e.g., the appendix in [1]) there exist monic univariate polynomials $g_n(x) = x^n + \cdots$, $t_n(y) = y^n + \cdots$ such that setting

$$\xi_n := \max_{x \in I \setminus T_1} |g_n(x)|, \qquad \eta_n := \max_{y \in I \setminus T_2} |t_n(y)|, \tag{1}$$

$$\xi_n, \eta_n \leqslant 2^{-n+1}, \qquad n \in \mathbb{N}$$

we have for some $\beta > 1$ and $n_1 \in \mathbb{N}$

$$\xi_n, \eta_n \leqslant (2\beta)^{-n}, \qquad n \geqslant n_1. \tag{2}$$

Set $\lambda_n := E(f, \Omega_n)$, $n \in \mathbb{N}$. Since $\lambda_n \downarrow 0$ as $n \to \infty$ by a standard argument (see [2, p. 4]) for some infinite subsequence $T \subset \mathbb{N}$

$$\frac{\lambda_n - \lambda_{n+1}}{\lambda_n + \lambda_{n+1}} \geqslant \frac{1}{n^2}, \qquad n \in T. \tag{3}$$

Consider now arbitrary $p_n \in B(f, \Omega_n)$, $n \in \mathbb{N}$. Then it is known (see [3, p. 14]) that there exist $m \in \mathbb{N}$, $\mathbf{z}_k = (x_k, y_k) \in I^2$ and $c_k \neq 0$ $(1 \leqslant k \leqslant m)$ such that

$$(f - p_n)(\mathbf{z}_k) = E(f, \Omega_n) \operatorname{sgn} c_k, \qquad 1 \leqslant k \leqslant m, \tag{4}$$

$$\sum_{k=1}^{m} c_k \, p(\mathbf{z}_k) = 0, \qquad p \in P(\Omega_n). \tag{5}$$

Setting

$$p_{n+1}^* := \frac{p_{n+1} - p_n}{\lambda_{n+1} + \lambda_n} \in P(\Omega_{n+1}),$$

we clearly have $\|p_{n+1}^*\| \leqslant 1$. Moreover (4) and (3) yield

$$\begin{aligned} p_{n+1}^*(\mathbf{z}_k) \operatorname{sgn} c_k &= \frac{(f - p_n)(\mathbf{z}_k) - (f - p_{n+1})(\mathbf{z}_k)}{\lambda_n + \lambda_{n+1}} \operatorname{sgn} c_k \\ &\geqslant \frac{\lambda_n - \lambda_{n+1}}{\lambda_n + \lambda_{n+1}} \geqslant \frac{1}{n^2}, \qquad n \in T, \quad 1 \leqslant k \leqslant m. \end{aligned} \tag{6}$$

In addition, with some $p_n^{**} \in P(\Omega_n)$ we have

$$p_{n+1}^*(\mathbf{z}) = \sum_{\mathbf{r} \in \Omega_{n+1} \setminus \Omega_n} a_{\mathbf{r}}^* \mathbf{z}^{\mathbf{r}} + p_n^{**}(\mathbf{z}). \tag{7}$$

Properties (i)–(ii) of Ω_n yield that $\#\{\Omega_{n+1}\backslash\Omega_n\}\leqslant cn$ with an absolute constant $c>0$, and, in addition, the lemma is applicable to every $a^*_{\mathbf{r}}$ in (7). Hence, whenever $\mathbf{r}=(i,j)\in\Omega_{n+1}\backslash\Omega_n$

$$|a^*_{\mathbf{r}}|\leqslant 2^{i+j-1}\|p^*_{n+1}\|\leqslant 2^{i+j-1}. \tag{8}$$

Consider now the polynomial

$$\tilde{p}_{n+1}(x,y)=p^*_{n+1}(x,y)-\sum_{\mathbf{r}=(i,j)\in\Omega_{n+1}\backslash\Omega_n} a^*_{\mathbf{r}}g_i(x)t_j(y), \tag{9}$$

where g_i, t_j are monic univariate polynomials satisfying (1) and (2). Then properties (i)–(ii) imply that $\tilde{p}_{n+1}\in P(\Omega_n)$, i.e., by (5)

$$\sum_{k=1}^{m} c_k\tilde{p}_{n+1}(\mathbf{z}_k)=0. \tag{10}$$

Furthermore, using that $x\in I\backslash T_1$, $y\in I\backslash T_2$ for every $\mathbf{z}=(x,y)\in\mathbf{A}(f,p_n)$ and $n\geqslant n_0$ we have by (9), (8) and (1)

$$|\tilde{p}_{n+1}-p^*_{n+1}|(\mathbf{z})\leqslant\sum_{(i,j)\in\Omega_{n+1}\backslash\Omega_n} 2^{i+j-1}\xi_i\eta_j,\qquad \mathbf{z}\in\mathbf{A}(f,p_n). \tag{11}$$

Setting $m_n:=\min\{i+j\colon (i,j)\in\mathbb{N}^2\backslash\Omega_n\}$ we clearly have that $i+j\geqslant m_n$ for every $(i,j)\in\Omega_{n+1}\backslash\Omega_n$. Recall that by property (iii) of Ω_n, $m_n/\log n\to\infty$ $(n\to\infty)$. Hence by (1) and (2) for n large enough

$$\xi_i\eta_j\leqslant 2^{-i-j+1}\beta^{-m_n/2},\qquad (i,j)\in\Omega_{n+1}\backslash\Omega_n.$$

Using this estimate in (11) yields for every $\mathbf{z}\in\mathbf{A}(f,p_n)$,

$$|\tilde{p}_{n+1}-p^*_{n+1}|(\mathbf{z})\leqslant\#\{\Omega_{n+1}\backslash\Omega_n\}\beta^{-m_n/2}\leqslant cn\beta^{-m_n/2}.$$

Finally, combining the last estimate with (6) we obtain for $n\in T$ large enough

$$\begin{aligned}\tilde{p}_{n+1}(\mathbf{z}_k)\operatorname{sgn}c_k&\geqslant p^*_{n+1}(\mathbf{z}_k)\operatorname{sgn}c_k-|p^*_{n+1}-\tilde{p}_{n+1}|(\mathbf{z}_k)\\&\geqslant\frac{1}{n^2}-cn\beta^{-m_n/2},\qquad 1\leqslant k\leqslant m.\end{aligned}\tag{12}$$

Since $\beta>1$ and $m_n/\log n\to\infty$ $(n\to\infty)$ it follows from (12) that $\tilde{p}_{n+1}(\mathbf{z}_k)$ $\operatorname{sgn}c_k$ is positive for every $1\leqslant k\leqslant m$ and $n\in T$ large enough. But this clearly contradicts (10). The theorem is proved. ∎

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